Monochromatic Hamiltonian t-tight Berge-cycles in hypergraphs

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Abstract

In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge E_i of $\mathcal{H}$ that contains these t vertices and the edges E_i are all distinct for $i, 1 \leq i \leq \ell$ where $\ell + j \equiv j$. For $t = 2$ we get the classical Berge-cycle and for $t = r$ we get the so-called tight cycle. In this note we formulate the following conjecture. For any fixed $2 \leq c, t \leq r$ satisfying $c+t \leq r+1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices, with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle. We prove some partial results about this conjecture and we show that if true the conjecture is best possible.

1 Introduction

The investigations of Turán type problems for paths and cycles of graphs were started by Erdős and Gallai in [3]. The corresponding Ramsey problems have

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been looked at some years later first in [6] and then later in [4], [5], [8], [12] and [14].

There are several possibilities to define paths and cycles in hypergraphs. In this paper we address the case of the *Berge-cycle*; probably it is the earliest definition of a cycle in hypergraphs in the book of Berge [1]. Turán type problems for Berge-paths and Berge-cycles of hypergraphs appeared perhaps first in [2]. Other types of hypergraph cycles, *loose* and *tight*, have been studied in [11], [13] and [15]. The investigations of the corresponding Ramsey problems started quite recently with [9] and [10] where Ramsey numbers of loose and tight cycles have been determined asymptotically for two colors and for 3-uniform hypergraphs.

Let $\mathcal{H}$ be an r-uniform hypergraph (some r-element subsets of a set). Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices. In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge E_i of $\mathcal{H}$ that contains these t vertices and the edges E_i are all distinct for $i, 1 \leq i \leq \ell$ where $\ell + j \equiv j$. We will denote by $E(C_\ell^{(r,t)})$ the set of these edges E_i used on the cycle. For $t = 2$ we get Berge-cycles and for $t = r$ we get the tight cycle. When the uniformity is clearly understood we may simply write $C_\ell^{(t)}$ for $C_\ell^{(r,t)}$ or just C_ℓ. $R_c(C_\ell^{(r,t)})$ will denote the Ramsey number of the r-uniform t-tight ℓ cycle using c colors. A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this note, continuing the investigations from [7], we study Hamiltonian Berge-cycles in hypergraphs. Thinking in terms of graphs, such an attempt seems strange, since in many 2-coloring of K_n there are no monochromatic Hamiltonian cycles. For example, if each edge incident to a fixed vertex is red and the other edges are blue, there is no monochromatic Hamiltonian cycle. However, from the nature of Berge-cycles, this example does not carry over to hypergraphs, if $K_n^{(3)}$ is colored this way, there is a red Hamiltonian Berge-cycle (for $n \geq 5$).

In [7] monochromatic Hamiltonian (2-tight) Berge-cycles were studied and the following conjecture was formulated. Assume that $r > 1$ is fixed and n is sufficiently large. Then every $(r-1)$-coloring of $K_n^{(r)}$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle. The conjecture was proved for $r = 3$. For general r, the statement was proved for sufficiently large n with $\lfloor \frac{r-1}{2} \rfloor$ colors instead of $r - 1$ colors. In this note we look at monochromatic Hamiltonian t-tight Berge-cycles and we generalize the above conjecture in the following way.

Conjecture 1. *For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

We will prove that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c+t > r+1$ the statement is not true.

Theorem 2. *For any fixed $2 \leq c, t \leq r$ satisfying $c+t > r+1$ and sufficiently large n, there is a coloring of the edges of $K_n^{(r)}$ with c colors, such that the longest monochromatic t-tight Berge-cycle has length at most $\lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$.*

We know that Conjecture 1 is true for $c = t = 2$ and $r = 3$, see [7]. It has also been proved in [7] that Conjecture 1 is *asymptotically* true for $c = 3$, $t = 2$ and $r = 4$. For the symmetrical case, $c = 2$, $t = 3$, we were able to prove only the following weaker but *sharp* result.

Theorem 3. *For any $n \geq 7$, if the edges of $K_n^{(5)}$ are colored with two colors, then there exists a monochromatic Hamiltonian 3-tight Berge-cycle.*

Note that Conjecture 1 would imply the same statement with $r = 4$ instead of $r = 5$, however, at this point we were unable to prove the statement for $r = 4$.

Similarly as in [7], for general r we were able to obtain only the following weaker result, where essentially we replace the sum $c + t$ with the product ct.

Theorem 4. *For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and $n \geq 2(t+1)rc^2$, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In Section 2 we give the simple construction for Theorem 2. In Sections 3 and 4 we present the proofs of Theorems 3 and 4.

2 The construction

Proof. (of Theorem 2)

Let $A_1, \ldots, A_{c-1}$ be disjoint vertex sets of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$. The r-edges not containing a vertex from A_1 are colored with color 1. The r-edges that are not colored yet and do not contain a vertex from A_2 are colored with color 2. We continue in this fashion. Finally the r-edges that are not colored yet with colors $1, \ldots, c-2$ and do not contain a vertex from A_{c-1} are colored with color $c-1$. The r-edges that contain a vertex from all $c-1$ sets $A_1, \ldots, A_{c-1}$ (if such r-edges exist) get color c. We claim that in this c-coloring of the edges of $K_n^{(r)}$ the longest monochromatic t-tight Berge-cycle has length $\leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$. This is certainly true for Berge-cycles in color i for $1 \leq i \leq c-1$, since the subhypergraph induced by the edges in color i leaves out A_i (a set of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$) completely. Finally, note that in a t-tight Berge-cycle in color c (if such a cycle exists) from t ($> r - c + 1$) consecutive vertices on the cycle at least one has to come from $A_1 \cup \ldots \cup A_{c-1}$ and thus the cycle has length at most

$$t(c-1)\lfloor \frac{n}{t(c-1)+1} \rfloor \leq \frac{t(c-1)n}{t(c-1)+1} \leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil.$$

□

3 3-tight 5-uniform Berge-cycles

Lemma 5. *If the edges of $K_7^{(5)}$ are colored with 2 colors, there exists a monochromatic Hamiltonian 3-tight Berge-cycle.*

Proof. We first remark that the hypergraph $K_7^{(5)}$ contains 21 edges, that each pair is contained in exactly 10 edges, and each triple is contained in exactly 6 edges.

Let us consider a coloring of the edges of $K_7^{(5)}$ in two colors, blue and red. We will first consider two favorable cases, when the edges containing a pair or a triple of vertices are mostly colored with the same color.

Case 1: Suppose that there exists a pair of vertices (for instance $\{0, 4\}$) contained in less than 3 edges of a color (for instance blue); that is it is contained in at least 8 red edges. Without loss of generality, we can assume that if there are blue edges containing $\{0, 4\}$, one is $(0, 1, 2, 3, 4)$ and possibly a second one is either $(0, 1, 4, 5, 6)$ or $(0, 1, 2, 4, 5)$.

Let us consider the cycle $(0, 6, 2, 3, 4, 5, 1)$. In Table 1, we give a choice of a red edge for each triple of consecutive vertices of this cycle, all distinct.

Table 1: Choice of a red edge for each triple for Lemma 5 Case 1.

$\{0, 6, 2\}$:	(0, 2, 4,5,6)
$\{6, 2, 3\}$:	(0, 2,3,4, 6)
$\{2, 3, 4\}$:	(0, 2,3,4,5)
$\{3, 4, 5\}$:	(0, 3,4,5,6)
$\{4, 5, 1\}$:	(0,1, 3,4,5)
$\{5, 1, 0\}$:	(0,1,2, 4,5) or (0,1, 4,5,6)
$\{1, 0, 6\}$:	(0,1, 3,4, 6)

Case 2: Suppose now that every pair of vertices is contained in at least 3 edges of each color. Suppose that for some triple of vertices, say $\{0, 1, 2\}$, all the 6 edges containing it are of the same color, for instance red.

Consider the pair $\{3, 6\}$, at least three red edges contains it. One of them is $(0, 1, 2, 3, 6)$, let $(3, 6, \alpha, \beta, \gamma)$ be another one. Necessarily, $\{\alpha, \beta, \gamma\} \cap \{0, 1, 2\} \neq \emptyset$, so we can suppose without loss of generality $\gamma = 2$.

We give in Table 2 a choice of a red edge for each triple of consecutive vertices for the cycle $(0, 3, 6, 2, 4, 1, 5)$. All these edges are obviously distinct, except perhaps for $(2, 3, 6, \alpha, \beta)$. Yet this edge may be equal only to $(0, 1, 2, 3, 6)$, and we chose them to be different. So this cycle with this choice of edges forms a red Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$.

Table 2: Choice of a red edge for each triple for Lemma 5 Case 2.

$\{0,3,6\}$:	(0,1,2,3, 6)
$\{3,6,2\}$:	(2,3, 6,α,β)
$\{6,2,4\}$:	(0,1,2, 4, 6)
$\{2,4,1\}$:	(0,1,2,3,4)
$\{4,1,5\}$:	(0,1,2, 4,5)
$\{1,5,0\}$:	(0,1,2, 5,6)
$\{5,0,3\}$:	(0,1,2,3, 5)

Case 3: Finally, we can assume that every pair of vertices is contained in 3 edges of each color and that every triple of vertices is contained in an edge of each color.

The hypergraph $K_7^{(5)}$ contains 21 edges, so there must be 11 edges of the same color, suppose red. By the pigeonhole principle, we will prove that there must exist a triple that is contained in at least 4 red edges. Each red edge contains exactly $\binom{5}{3} = 10$ distinct triples, this makes at least 110 pairs $\{e, f\}$ such that e is a red edge and f is a triple with $f \subset e$. There are exactly $\binom{7}{3} = 35$ triples, now $\frac{110}{35} > 3$, so there exists a triple that is contained in at least 4 red edges.

Let the triple $\{0, 1, 2\}$ be contained in at least 4 red edges. It is also contained in a blue edge, suppose $(0, 1, 2, 4, 5)$. If there is a second blue edge containing $\{0, 1, 2\}$, we assume without loss of generality that it is either $(0, 1, 2, 3, 6)$ or $(0, 1, 2, 4, 6)$. Consider the pair $\{4, 5\}$; it is contained in at least 3 red edges: e_1, e_2 and e_3. Since none are equal to $(0, 1, 2, 4, 5)$, they all contain the vertex 3 or 6, maybe both. Moreover, since both triples $\{3, 4, 5\}$ and $\{4, 5, 6\}$ are contained in a red edge, then at least one contains 3 and one contains 6. Suppose e_1 contains 3 and e_3 contains 6, e_2 contains either 3 or 6. We consider 3 subcases:

1. If $(0, 1, 2, 4, 6)$ is red:
   In this case, since the edge $(0, 1, 2, 3, 4)$ is also red, we may assume without loss of generality that e_2 contains 6. The edge e_3 contains either 0, 1, or 2; by symmetry, suppose it is 0. We form the cycle $(0, 1, 2, 3, 4, 5, 6)$ with the choice of edges given in table 3, first column.

2. If $(0, 1, 2, 4, 6)$ is blue and e_2 contains 6:
   The edge e_3 necessarily contains a vertex among 0, 1 and 2, suppose it is 0. Then, we form the cycle $(0, 1, 2, 3, 4, 5, 6)$ with the choice of edges given in table 3, second column.

3. If $(0, 1, 2, 4, 6)$ is blue and e_2 contains 3: The edge e_1 necessarily contains a vertex among 0, 1 and 2, suppose it is 2. Then, we form the cycle $(0, 1, 2, 3, 4, 5, 6)$ with the choice of edges given in table 3, third column.

Thus in every case, we managed to build a monochromatic Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$. $\square$

Table 3: Choice of a red edge for each triple for Lemma 5 Case 3.

triple :	Subcase 1	Subcase 2	Subcase 3
$\{0,1,2\}$:	(0,1,2, 5,6)	(0,1,2, 5,6)	(0,1,2,3, 5)
$\{1,2,3\}$:	(0,1,2,3, 5)	(0,1,2,3, 5)	(0,1,2,3,4)
$\{2,3,4\}$:	(0,1,2,3,4)	(0,1,2,3,4)	e_1
$\{3,4,5\}$:	e_1	e_1	e_2
$\{4,5,6\}$:	e_2	e_2	e_3
$\{5,6,0\}$:	e_3	e_3	(0,1,2, 5,6)
$\{6,0,1\}$:	(0,1,2, 4, 6)	(0,1,2,3, 6)	(0,1,2,3, 6)

Proof. (of Theorem 3)

Consider the complete hypergraph $\mathcal{H} = K_n^{(5)}$ whose edges are 2-colored. We will proceed by induction on n, its number of vertices. Lemma 5 establishes the base case for $n = 7$. Let $n \geq 8$. Suppose the result is true for $n - 1$.

Let a be a vertex of $\mathcal{H}$. By the induction hypothesis, the induced subgraph of $\mathcal{H}$ on all its vertices except a has a monochromatic Hamiltonian 5-uniform 3-tight Berge-cycle C. Say its color is *carmine*, the other color being *azure*. Let us name its vertices $\{1, 2, \ldots, n-1\}$ in the order they appear in the cycle.

In the following, we will give a color to any pair $\{x, y\}$ of vertices of $V \setminus \{a\}$, depending on the color of the edges containing x, y and a. We will say a pair $\{x, y\}$ is *red* if all the edges containing x, y and a are carmine, except perhaps one. We will say a pair $\{x, y\}$ is *blue* if all the edges containing x, y and a are azure, except perhaps one. Otherwise, we will say a pair is *green*, meaning at least 2 edges containing x, y and a are carmine and at least 2 are azure.

Remark that if a pair containing x is red, then no pairs containing x can be blue, and vice versa. To prove it, suppose a pair $\{x, y\}$ is red while a pair $\{x, z\}$ is blue. Take three vertices $u, v, w \notin \{a, x, y, z\}$. Consider the three edges (a, x, y, z, u), (a, x, y, z, v), and (a, x, y, z, w). Two of them have the same color, say carmine, then $\{x, z\}$ cannot be blue, and if the color is azure, $\{x, y\}$ cannot be red.

Suppose first that there exists a $1 \leq i \leq n-1$ such that the pairs $\{i, i+1\}$, $\{i+1, i+2\}$, and $\{i+2, i+3\}$ (with $n - 1 + j \equiv j$) are green or red. For notation convenience, suppose $i = 1$. We claim that there is a choice of edges such that $(1, 2, a, 3, 4, \ldots n-1)$ is a 3-tight monochromatic carmine Hamiltonian cycle. Let us define such a choice of edges. For any $3 \leq j \leq n-1$, choose for the set $\{j, j+1, j+2\}$ the corresponding edge in C. Three edges still have to be found, corresponding to the sets $\{1, 2, a\}$, $\{2, a, 3\}$ and $\{a, 3, 4\}$. For these three sets, we will choose edges containing a, that are therefore different from the edges we took before.

Since the pairs $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$ are green or red, there are at least two carmine edges containing each of the sets $\{a, 1, 2\}$, $\{a, 2, 3\}$ and $\{a, 3, 4\}$.

If the edge $(1, 2, 3, 4, a)$ is carmine, take it for the set $\{2, a, 3\}$. Now choose any other carmine edge for $\{1, 2, a\}$ and $\{a, 3, 4\}$. There exist such edges since $\{1, 2\}$ and $\{3, 4\}$ are green or red, and they are distinct since different from $(1, 2, 3, 4, a)$. Otherwise, take any suiting carmine edge for $\{2, a, 3\}$, and different carmine edges for $\{1, 2, a\}$ and $\{a, 3, 4\}$. All these edges exist since $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$ are green or red, and the edge for $\{1, 2, a\}$ and $\{a, 3, 4\}$ are different or it would be $(1, 2, 3, 4, a)$, which is azure.

Now we can suppose that for any $1 \le i \le n-1$, $\{i, i+1\}$, $\{i+1, i+2\}$, or $\{i+2, i+3\}$ is blue. Since most edges are now blue, we are tempted to try to form a cycle of color azure. We will still form a carmine cycle in the following case.

Suppose there exists a vertex $1 \le i \le n-1$, such that the edges $(a, i, i+1, i+2, i+3)$, $(a, i, i+1, i+2, i+4)$ and $(a, i, i+1, i+2, i+5)$ are carmine. Then to form a carmine cycle, we insert a between $i+1$ and $i+2$. We get the cycle $(1, 2, \dots, i, i+1, a, i+2, i+3, \dots, n-1)$. For $\{i, i+1, a\}$, we use the edge $(a, i, i+1, i+2, i+5)$, for $\{i+1, a, i+2\}$, the edge $(a, i, i+1, i+2, i+4)$, for $\{a, i+2, i+3\}$, the edge $(a, i, i+1, i+2, i+3)$, and for all the other triples, we use the corresponding edge of C.

We finally can assume otherwise that for any $1 \le i \le n-1$, one of the edges $(a, i, i+1, i+2, i+3)$, $(a, i, i+1, i+2, i+4)$ and $(a, i, i+1, i+2, i+5)$ is azure. Then using this edge for the set $\{i, i+1, i+2\}$, we form an azure cycle C' $\{1, 2, \dots n\}$ not containing a. All the edges we used are distinct since $n-1 > 6$. Let us choose a blue pair of consecutive vertices in the cycle. Without loss of generality, suppose the pair is $\{2, 3\}$. We will insert the vertex a between 2 and 3 in the cycle C'. Most edges may remain unchanged. For the set $\{1, 2, a\}$, we can use the edge of C' formerly used for $\{1, 2, 3\}$ which contains a by construction of C'. Likewise, we can use for $\{a, 3, 4\}$ the edge of C' formerly used for $\{2, 3, 4\}$. We only have to find an edge for $\{2, a, 3\}$. Since $\{2, 3\}$ is blue, either $(2, a, 3, 5, 6)$ or $(2, a, 3, 5, 7)$ is azure, and they both are distinct from any edge of C'. So we can find among these two an edge for $\{2, a, 3\}$, and we get a monochromatic Hamiltonian 3-tight Berge-cycle. $\square$

4 Proof of Theorem 4

Proof. (of Theorem 4)

We follow the method of [7]. For the sake of completeness we give the details. We first prove the following lemma.

Lemma 6. *Let k and $t \ge 2$ be fixed positive integers and let $n > 2(t+1)tk$. Then a $(t+1)$-uniform hypergraph $\mathcal{H}$ of order n with at least $\binom{n}{t+1} - kn$ edges has a Hamiltonian t-tight Berge-cycle.*

Proof. By averaging there exists a vertex $x \in V(\mathcal{H})$ contained in at least $\binom{n-1}{t} - (t+1)k$ edges of $\mathcal{H}$. Thus apart from at most $(t+1)k$ exceptional sets all subsets of size t on the remaining $n-1$ vertices form an edge of $\mathcal{H}$ together with x. Let us denote the union of the vertices in the exceptional subsets by U. Thus $|U| \leq (t+1)kt$. Take a cyclic permutation on the remaining vertices where two vertices from U are never neighbors. Since $n > 2(t+1)tk$, this is possible. But then this cyclic permutation is actually a t-tight Berge-cycle, i.e. $C_{n-1}^{(t+1,t)}$. Indeed, any set of t consecutive vertices on the cycle contains a non-exceptional vertex and thus it forms an edge with x. Furthermore, since $n > 2(t+1)tk$, there must be two non-exceptional vertices, denoted by x_1 and y_1, that are neighbors on the cycle. Consider the $2t$ consecutive vertices along the cycle that include x_1 and y_1 in the middle, and denote these vertices by $x_t, \dots, x_1, y_1, \dots, y_t$. Consider also a vertex z along the cycle that is not among these $2t$ vertices. We claim that x can be inserted between x_1 and y_1 on the cycle and thus giving a Hamiltonian t-tight Berge-cycle in $\mathcal{H}$. Indeed, for those sets of t consecutive vertices which do not include x, we can add x to get the required edge E_i. If a set of t consecutive vertices includes x, then it also must include either x_1 or y_1 (or maybe both), i.e. a non-exceptional vertex. But then we can add z to get the required edge. It is easy to check that all the used edges are distinct. $\square$

For $S \subseteq V(K_n^{(g)}), |S| < g$, let $E_S = \{e | e \in E(K_n^{(g)})$ with $S \subseteq e\}$, the set of edges containing S. Thus $|E_S| = \binom{n-|S|}{g-|S|}$. It is enough to prove Theorem 4 for $r = ct+1$. Indeed, for $r > ct+1$, one can have a color transfer by any injection of the $(ct+1)$-element subsets of the n vertices into their r-element supersets ($n \geq 2r$ is ensured). Then Theorem 4 will easily follow from the following stronger theorem.

Theorem 7. *Let $c, t \geq 2$ and let $n \geq 2(t+1)tc^2$. Furthermore let $S \subseteq V(K_n^{(ct+1)})$ such that S is of order divisible by t (possibly empty) with $|S| \leq (c-1)t$. Set $u = c - \frac{|S|}{t}$ (≥ 1). Color $m \geq f(n,u,S)$ edges of E_S with u colors. If $f(n,u,S) \geq \binom{n-|S|}{ct+1-|S|} - (c-u)(n+t) > 0$, then E_S contains a monochromatic Hamiltonian t-tight Berge-cycle.*

Proof. Let $F_S \subseteq E_S$, $|F_S| = m$, be the set of colored edges in E_S. Fix $t \geq 2$. The proof will be by induction on $u, 1 \leq u \leq c$. If $u = 1$, then $|S| = (c-1)t$ so that $\binom{n-|S|}{ct+1-|S|} - (c-1)(n+t) = \binom{n-(c-1)t}{t+1} - (c-1)(n+t) \geq \binom{n-(c-1)t}{t+1} - c(n-(c-1)t)$ when $n \geq tc^2$. Define the $(t+1)$-uniform hypergraph $\mathcal{H}_S$ with $V(\mathcal{H}_S) = V(K_n^{(ct+1)}) \setminus S$ and $E(\mathcal{H}_S) = \{e \setminus S \mid e \in F_S\}$. Therefore since $n-(c-1)t > 2(t+1)tc$ by Lemma 6 $\mathcal{H}_S$ contains a Hamiltonian t-tight Berge-cycle $C_{n-(c-1)t}^{(t+1,t)}$. Then we get the corresponding t-tight Berge-cycle $C_{n-(c-1)t}^{(ct+1,t)}$ in E_S. But each edge of E_S contains S and only $n-(c-1)t$ edges are used on this $C_{n-(c-1)t}^{(ct+1,t)}$ so that it is easy to insert all of S in place of any edge of $C_{n-(c-1)t}^{(ct+1,t)}$ giving the monochromatic $C_n^{(ct+1,t)}$. Indeed, insert all the vertices of S in arbitrary order between two consecutive vertices on the cycle. Consider a set T of t consecutive vertices on the new cycle. If T does not contain a vertex from S, then we can use the edge E_i from $E(C_{n-(c-1)t}^{(ct+1,t)})$. If T does have at least one vertex from S, then it has at most $(t-1)$ vertices outside S, and thus at least $ct+1-|S|-(t-1)=2$ more vertices are "free", so in E_S the number of edges containing T that we can still use (not missing or not used on the cycle yet) is at least

$$\binom{n-|S\cup T|}{2}-(c+1)(n-(c-1)t)\geq$$

$$\geq \frac{(n-ct)^2}{2}-(c+1)(n-(c-1)t).$$

Thus we can select a distinct edge E_i for each such T if

$$\frac{(n-ct)^2}{2}-(c+1)(n-(c-1)t)\geq ct,$$

which is certainly true for $n\geq 2(t+1)tc^2$.

Therefore assume the theorem holds for $u-1$ colors with $c\geq u\geq 2$ and color the m edges of E_S by u colors, $m\geq f(n,u,S)\geq \binom{n-|S|}{ct+1-|S|}-(c-u)(n+t)>0, |S|=(c-u)t$. In F_S select a maximum length monochromatic t-tight Berge-cycle. Suppose first that this is $C_\ell^{(ct+1,t)}=(z_1,z_2,\dots,z_\ell)$ in color 1, with $2t-2\leq \ell<n$. We will handle the case $\ell<2t-2$ later. Let $z\in V(K_n^{(ct+1)})\setminus V(C_\ell^{(ct+1,t)})$. Consider the vertices $\{z_1,z_2,\dots,z_{2t-2}\}$ (using $2t-2\leq\ell$) and the t subsets $T_1,\dots,T_t$ consisting of $t-1$ consecutive vertices in this interval. If for each $i, 1\leq i\leq t$ the set $T_i\cup\{z\}$ is contained in at least t distinct edges in $E_S\setminus E(C_\ell^{(ct+1,t)})$ in color 1, then clearly we could insert z into the cycle between z_{t-1} and z_t, a contradiction. Hence we may assume that for some T_i (say T_1 without loss of generality) apart from at most $(c-u)(n+t)+t$ exceptional edges all edges in $E_{S\cup T_1\cup\{z\}}\setminus E(C_\ell^{(ct+1,t)})$ are in color $2,3,\dots,u$.

Assume now the second case, $\ell<2t-2$. Consider arbitrary vertices $\{z_1,z_2,\dots,z_{2t}\}\in V(K_n^{(ct+1)})\setminus S$ in a cyclic order and the $2t$ subsets $T_1,\dots,T_{2t}$ consisting of t consecutive vertices in this cyclic order. If for each $i, 1\leq i\leq 2t$ the set T_i is contained in at least $2t$ distinct edges in E_S in color 1, then we would have a t-tight Berge-cycle of length $2t$ in color 1 in F_S, a contradiction. Hence we may assume that for some T_i (say T_1 without loss of generality) apart from at most $(c-u)(n+t)+2t$ exceptional edges all edges in $E_{S\cup T_1}$ are in color $2,3,\dots,u$.

Let S' be any set of $|S|+t=(c-u+1)t$ vertices containing $S\cup T_1\cup\{z\}$ in the first case and $S\cup T_1$ in the second case. Thus in both cases at least $|E_{S'}|-(c-u+1)(n+t)$ edges of $E_{S'}$ are colored by at most $u-1$ colors. But $f(n,u-1,S')\geq |E_{S'}|-(c-u+1)(n+t)=\binom{n-(|S|+t)}{ct+1-(|S|+t)}-(c-(u-1))(n+t)>0, 1\leq u-1=c-\frac{|S'|}{t}$, and $|S'|=(c-u+1)t$, so by the induction assumption $E_{S'}$ contains a monochromatic Hamiltonian t-tight Berge-cycle, $C_n^{(ct+1,t)}$, contradicting the assumption that E_S contains no monochromatic $C_n^{(ct+1,t)}$. Therefore for any $u, 1\leq u\leq c$, E_S contains a monochromatic $C_n^{(ct+1,t)}$. □

Now the proof of Theorem 4 is concluded by applying Theorem 7 with $S=\emptyset$. □

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