WPI-CS-TR-15-02 September 2015

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by

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Computer Science
Technical Report
Series

WORCESTER POLYTECHNIC INSTITUTE

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A quantitative version of the Blow-up Lemma

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Abstract

In this paper we give a quantitative version of the Blow-up Lemma.

1 Introduction

1.1 Notations and definitions

All graphs are simple, that is, they have no loops or multiple edges. $v(G)$ is the number of vertices in G (order), $e(G)$ is the number of edges in G (size). $deg(v)$ (or $deg_G(v)$) is the degree of vertex v (within the graph G), and $deg(v, Y)$ (or $deg_G(v, Y)$) is the number of neighbors of v in Y. $\delta(G)$ and $\Delta(G)$ are the minimum degree and the maximum degree of G. $N(x)$ (or $N_G(x)$) is the set of neighbors of the vertex x, and $e(X, Y)$ is the number of edges between X and Y. A bipartite graph G with color-classes A and B and edges E will sometimes be written as $G = (A, B, E)$. For disjoint X, Y, we define the **density**

$$d(X,Y) = \frac{e(X,Y)}{|X| \cdot |Y|}.$$

*Research supported in part by OTKA Grant No. K104343.

The density of a bipartite graph $G = (A, B, E)$ is the number

$$d(G) = d(A, B) = \frac{|E|}{|A| \cdot |B|}\,.$$

For two disjoint subsets A, B of $V(G)$, the bipartite graph with vertex set $A \cup B$ which has all the edges of G with one endpoint in A and the other in B is called the pair (A, B).
A pair (A, B) is ε-**regular** if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

A pair (A, B) is (ε, d, δ)-**super-regular** if it is ε-regular with density at least d and furthermore,

$$deg(a) \geq \delta|B| \quad \text{for all} \quad a \in A,$$
$$\text{and} \quad deg(b) \geq \delta|A| \quad \text{for all} \quad b \in B.$$

H is **embeddable** into G if G has a subgraph isomorphic to H, that is, if there is a one-to-one map (injection) $\varphi : V(H) \to V(G)$ such that $\{x, y\} \in E(H)$ implies $\{\varphi(x), \varphi(y)\} \in E(G)$.

1.2 A quantitative version of the Blow-up Lemma

The Blow-up Lemma [8, 9] has been a successful tool in extremal graph theory. There are now at least four new proofs for the Blow-up Lemma since the original appeared; an algorithmic proof [9], a hypergraph-packing approach [17], a proof based on counting perfect matchings in (Szemerédi-) regular graphs [16], and its constructive version in [18]. Very recently the Blow-up Lemma has been generalized to hypergraphs by Keevash [6] and to d-arrangeable graphs by Böttcher, Kohayakawa, Taraz, and Würfl [3]. The Blow-up Lemma has been cited in close to 100 papers (see e.g. [1, 4, 7, 10, 11, 12, 13, 15, 16, 17, 18]). See also the discussion on the Regularity Lemma and the Blow-up Lemma on pages 803-804 in the Handbook of Graph Theory [2] or the survey paper [14].
In either of our proofs [8, 9], the dependence of the parameters was not computed explicitly (in particular the dependence of ε on the other parameters). In this paper we give a quantitative version, i.e. we compute explicitly the parameters.

Theorem 1 (A quantitative version of the Blow-up Lemma). *There exists an absolute constant C such that, given a graph R of order $r \geq 2$ and positive parameters*

d, δ, and Δ, for any $0 < \varepsilon < \left(\frac{\delta d^{\Delta}}{r\Delta}\right)^C$ the following holds. Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$, and a sparser graph G is constructed by replacing the edges of R with some (ε, d, δ)-super-regular pairs. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $R(N)$ then it is already embeddable into G.

Our proof is almost identical to the proof in [9]. Of course one difference is that we have to compute explicitly the dependence between the parameters. Furthermore, this is also a slight strengthening of the original statement as there can be a small number of exceptional vertices which may have smaller degrees (δ may be much smaller than d). We note that the recent "arrangeable" Blow-up Lemma [3] is also quantitative, but first of all the bound on ε is somewhat weaker and second it does not allow for the strengthening mentioned above. However, in a recent application [5] we needed precisely this strengthening. We believe that this quantitative version of the Blow-up Lemma will find other applications as well.
In Section 2 we give the embedding algorithm. In Section 3 we show that the algorithm is correct.

2 The algorithm

The main idea of the algorithm is the following. We embed the vertices of H one-by-one by following a greedy algorithm, which works smoothly until there is only a small proportion of H left, and then it may get stuck hopelessly. To avoid that, we will set aside a positive proportion of the vertices of H as buffer vertices. Most of these buffer vertices will be embedded only at the very end by using a König-Hall argument.

2.1 Preprocessing

We will assume that $|V(H)| = |V(G)| = |\cup_i V_i| = n = rN$. We will assume for simplicity, that the density of every super-regular pair in G is exactly d. This is not a significant restriction, otherwise we just have to put everywhere the actual density instead of d.
We will use the following parameters:

$$\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll d''' \ll d'' \ll d' \ll d,$$

where $a \ll b$ means that a is small enough compared to b. For example we can select the parameters in the following explicit way:

$$d' = \frac{\delta d^\Delta}{8r\Delta}, d'' = (d')^3 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^3, d''' = (d'')^2 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^6, \varepsilon''' = (d''')^2 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^{12},$$

$$\varepsilon'' = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^2 (d''')^2(\varepsilon''')^3 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^{50}, \varepsilon' = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^2 (d''')^2(\varepsilon'')^3 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^{164} \text{ and}$$

$$\varepsilon = (\varepsilon')^2 = \left(\frac{\delta d^\Delta}{8r\Delta}\right)^{328}.$$

For easier reading, we will mostly use the letter x for vertices of H, and the letter v for vertices of the host graph G.
Given an embedding of H into $R(N)$, it defines an *assignment*

$$\psi : V(H) \to \{V_1, V_2, \ldots, V_r\},$$

and we want to find an *embedding*

$$\varphi : V(H) \to V(G), \quad \varphi \text{ is one-to-one}$$

such that $\varphi(x) \in \psi(x)$ for all $x \in V(H)$. We will write $X_i = \psi^{-1}(V_i)$ for $i = 1, 2, \ldots, r$.
Before we start the algorithm, we order the vertices of H into a sequence $S = (x_1, x_2, \ldots, x_n)$ which is more or less, but not exactly, the order in which the vertices will be embedded (certain exceptional vertices will be brought forward). Let $m = rd'N$. For each i, choose a set B_i of $d'N$ vertices in X_i such that any two of these vertices are at a distance at least 3 in H. (This is possible, for H is a bounded degree graph.) These vertices $b_1, \ldots, b_m$ will be called the **buffer vertices** and they will be the last vertices in S.
The order S starts with the neighborhoods $N_H(b_1), N_H(b_2), \ldots, N_H(b_m)$. The length of this initial segment of S will be denoted by T_0. Thus $T_0 = \sum_{i=1}^{m} |N_H(b_i)| \leq \Delta m$.
The rest of S is an arbitrary ordering of the leftover vertices of H.

2.2 Sketch of the algorithm

In *Phase 1* of the algorithm we will embed the vertices in S one-by-one into G until all non-buffer vertices are embedded. For each x_j not embedded yet (including the buffer vertices) we keep track of an ever shrinking host set H_{t,x_j} that x_j is confined to at time t, and we only make a final choice for the location of x_j from H_{t,x_j} at time j. At time 0, H_{0,x_j} is the cluster that x_j is assigned to. For technical reasons we will also maintain another similar set, C_{t,x_j}, where we will ignore the possibility that some

vertices are occupied already. Z_t will denote the set of occupied vertices. Finally we will maintain a set Bad_t of exceptional pairs of vertices.
In *Phase 2*, we embed the leftover vertices by using a König-Hall type argument.

2.3 Embedding Algorithm

At time 0, set $C_{0,x} = H_{0,x} = \psi(x)$ for all $x \in V(H)$. Put $T_1 = d''n$.

Phase 1.
For $t \geq 1$, repeat the following steps.

Step 1 (Extending the embedding). We embed x_t. Consider the vertices in H_{t-1,x_t}. We will pick one of these vertices as the image $\varphi(x_t)$ by using the Selection Algorithm (described below in Section 2.4).

Step 2 (Updating). We set

$$Z_t = Z_{t-1} \cup \{\varphi(x_t)\},$$

and for each unembedded vertex y (i.e. the set of vertices $x_j, t < j \leq n$), set

$$C_{t,y} = \begin{cases} C_{t-1,y} \cap N_G(\varphi(x_t)) & \text{if } \{x_t, y\} \in E(H) \\ C_{t-1,y} & \text{otherwise,} \end{cases}$$

and

$$H_{t,y} = C_{t,y} \setminus Z_t.$$

We do not change the ordering at this step.

Step 3 (Exceptional vertices in G).
1. If $t \notin \{1, T_0\}$, then go to Step 4.
2. If $t = 1$, then we do the following (this is the part that is new compared to the proof in [9]). We find the 1st exceptional set (denoted by E_i^1) consisting of those exceptional vertices $v \in V_i$, $1 \leq i \leq r$ for which there exists a $j \neq i$ such that (V_i, V_j) is (ε, d, δ)-super-regular, yet

$$deg_G(v, V_j) < (d - \varepsilon)|V_j|.$$

(Note that $deg_G(v, V_j) \geq \delta|V_j|$ always holds by super-regularity.) ε-regularity implies that $|E_i^1| \leq r\varepsilon N$. We are going to change slightly the order of the vertices in S. We choose a set E_H^1 of nonbuffer vertices $x \in H$ of size $\sum_{i=1}^r |E_i^1|$ (more precisely $|E_i^1|$ vertices from X_i for all $1 \leq i \leq r$) such that they are at a distance at least 3 from each other. This is possible since H is a bounded degree graph and $\sum_{i=1}^{r}|E_i^1|$ is very small. We bring the vertices in E_H^1 forward, followed by the remaining vertices in the same relative order as before. For simplicity we keep the notation $(x_1, x_2, \ldots, x_n)$ for the resulting order. Furthermore, we slightly change the value of T_0 to $T_0 = |E_H^1| + \sum_{i=1}^{m} |N_H(b_i)|$.

3. If $t = T_0$, then we do the following. We find the 2nd exceptional set (denoted by E_i^2) consisting of those exceptional vertices $v \in V_i$, $1 \leq i \leq r$ for which v is not covered yet in the embedding and

$$|\{b : b \in B_i, v \in C_{t,b}\}| < d''|B_i|.$$

Once again we are going to change slightly the order of the remaining unembedded vertices in S. We choose a set E_H^2 of unembedded nonbuffer vertices $x \in H$ of size $\sum_{i=1}^{r}|E_i^2|$ (more precisely $|E_i^2|$ vertices from X_i for all $1 \leq i \leq r$) with

$$H_{t,x} = H_{0,x} \setminus \{\varphi(x_j) : j \leq t\} = \psi(x) \setminus \{\varphi(x_j) : j \leq t\}.$$

Thus in particular, if $x \in X_i$, then $E_i^2 \subset H_{t,x}$. Again we may choose the vertices in E_H as vertices in H that are at a distance at least 3 from each other and any of the vertices embedded so far. We are going to show later in the proof of correctness that this is possible since H is a bounded degree graph and $\sum_{i=1}^{r}|E_i^2|$ is very small as well. We bring the vertices in E_H^2 forward, followed by the remaining unembedded vertices in the same relative order as before. Again, for simplicity we keep the notation $(x_1, x_2, \ldots, x_n)$ for the resulting order.

Step 4 (Exceptional vertices in H).
1. If T_1 does not divide t, then go to Step 5.
2. If T_1 divides t, then we do the following. We find all exceptional unembedded vertices $y \in H$ such that $|H_{t,y}| \leq (d')^2 n$. Once again we slightly change the order of the remaining unembedded vertices in S. We bring these exceptional vertices forward (even if they are buffer vertices), followed by the non-exceptional vertices in the same relative order as before. Again for simplicity we still use the notation $(x_1, x_2, \ldots, x_n)$ for the new order. Note that it will follow from the proof, that if $t \leq 2T_0$, then we do not find any exceptional vertices in H, so we do not change the ordering at this step.

Step 5 - If there are no more unembedded non-buffer vertices left, then set $T = t$ and go to Phase 2, otherwise set $t \leftarrow t + 1$ and go back to Step 1.

Phase 2
Find a system of distinct representatives of the sets $H_{T,y}$ for all unembedded y (i.e. the set of vertices x_j, $T < j \leq n$).

2.4 Selection Algorithm

We distinguish two cases. Let $E_H = E_H^1 \cup E_H^2$.
Case 1. $x_t \notin E_H$.
We choose a vertex $v \in H_{t-1,x_t}$ as the image $\varphi(x_t)$ for which the following hold for all unembedded y with $\{x_t, y\} \in E(H)$,

$$(d-\varepsilon)|H_{t-1,y}| \le deg_G(v, H_{t-1,y}) \le (d+\varepsilon)|H_{t-1,y}|, \tag{1}$$

$$(d-\varepsilon)|C_{t-1,y}| \le deg_G(v, C_{t-1,y}) \le (d+\varepsilon)|C_{t-1,y}| \tag{2}$$

and

$$(d-\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}| \le deg_G(v, C_{t-1,y} \cap C_{t-1,y'}) \le (d+\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}|, \tag{3}$$

for at least a $(1-\varepsilon')$ proportion of the unembedded vertices y' with $\psi(y') = \psi(y)$ and $\{y, y'\} \notin \mathrm{Bad}_{t-1}$. Then we get Bad_t by taking the union of Bad_{t-1} and the set of all of those pairs $\{y, y'\}$ for which (3) does not hold for $v = \varphi(x_t)$, $C_{t-1,y}$ and $C_{t-1,y'}$. Thus note that we add at most $\Delta\varepsilon' N$ new pairs to Bad_t.
Case 2. $x_t \in E_H$.
If $x_t \in X_i \cap E_H^l$, $l = 1, 2$, then we choose an arbitrary vertex of E_i^l as $\varphi(x_t)$. Note that for all $y \in N_H(x_t)$, we have $C_{t-1,y} = \psi(y)$,

$$deg_G(\varphi(x_t), C_{t-1,y}) = deg_G(\varphi(x_t), \psi(y)) \ge \delta N = \delta |C_{t-1,y}|, \tag{4}$$

and

$$deg_G(\varphi(x_t), H_{t-1,y}) \ge deg_G(\varphi(x_t), \psi(y)) - T_0 - |E_H| \ge \delta N - 2\Delta r d' N \ge \frac{\delta}{2} N \tag{5}$$

(using our choice of parameters). Here we used super-regularity and the fact that $|E_H| \ll \Delta m$ which will be shown later (Lemma 3).

3 Proof of correctness

The following claims state that our algorithm finds a good embedding of H into G.

Claim 1. *Phase 1 always succeeds.*

Claim 2. *Phase 2 always succeeds.*

If at time t, S is a set of unembedded vertices $x \in H$ with $\psi(x) = V_i$ (here and throughout the proof when we talk about time t, we mean *after* Phase 1 is executed

for time t, so for example x_t is considered embedded at time t), then we define the bipartite graph U_t as follows. One color class is S, the other is V_i, and we have an edge between an $x \in S$ and a $v \in V_i$ whenever $v \in C_{t,x}$.
In the proofs of the above claims the following lemma will play a major role. First we prove the lemma for $t \leq T_0$, from this we deduce that $|E_H|$ is small, then we prove the lemma for $T_0 < t \leq T$.

Lemma 2. *We are given integers $1 \leq i \leq r$, $1 \leq t \leq T_0$ and a set $S \subset X_i$ of unembedded vertices at time t with $|S| \geq (d''')^2|X_i| = (d''')^2 N$. If we assume that Phase 1 succeeded for all time t' with $t' \leq t$, then apart from an exceptional set F of size at most $\varepsilon'' N$, for every vertex $v \in V_i$ we have the following*

$$deg_{U_t}(v) = |\{x : x \in S, v \in C_{t,x}\}| \geq (1-\varepsilon'')d(U_t)|S| \quad \left(\geq \frac{d^\Delta}{2}|S|\right).$$

Proof. In the proof of this lemma we will use the "defect form" of the Cauchy-Schwarz inequality (just as in the original proof of the Regularity Lemma [19]): if

$$\sum_{k=1}^{m} X_k = \frac{m}{n}\sum_{k=1}^{n} X_k + D \qquad (m \leq n)$$

then

$$\sum_{k=1}^{n} X_k^2 \geq \frac{1}{n}\left(\sum_{k=1}^{n} X_k\right)^2 + \frac{D^2 n}{m(n-m)}.$$

Assume indirectly that the statement in Lemma 2 is not true, that is, $|F| > \varepsilon'' N$. We take an $F_0 \subset F$ with $|F_0| = \varepsilon'' N$. Let us write $\nu(t,x)$ for the number of neighbors (in H) of x embedded by time t. Then in U_t using the left side of (2) we get

$$e(U_t) = d(U_t)|S||V_i| = \sum_{v \in V_i} deg_{U_t}(v) = \sum_{x \in S} deg_{U_t}(x)$$

$$= \sum_{x\in S}|C_{t,x}| \geq \sum_{x \in S}(d-\varepsilon)^{\nu(t,x)}N - \Delta r^2 \varepsilon N^2 \geq (d-\varepsilon)^\Delta |S|N - \Delta r^2\varepsilon N^2 \geq \frac{d^\Delta}{2}|S|N, \quad (6)$$

where the error term comes from the neighbors of elements of E_H^1 (we are yet to start the embedding of the vertices in E_H^2), since for them we cannot guarantee the same lower bound.
We also have

$$\sum_{x\in S}\sum_{x' \in S}|N_{U_t}(x) \cap N_{U_t}(x')| = \sum_{x\in S}\sum_{x'\in S}|C_{t,x} \cap C_{t,x'}|$$

$$\leq \sum_{x\in S}\sum_{x'\in S}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}N + |S|N + \Delta^2|S|N + 2\Delta r^2 \varepsilon |S| N^2 + 2\Delta\varepsilon' N^3$$

$$\leq \sum_{x\in S}\sum_{x'\in S}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}N + 5\Delta\varepsilon' N^3. \tag{7}$$

The error terms come from the following (x, x') pairs. For each such pair we estimate $|C_{t,x} \cap C_{t,x'}| \leq N$. The first error term comes from the pairs where $x = x'$. The second error term comes from those pairs (x, x') for which $N_H(x) \cap N_H(x') \neq \emptyset$. The number of these pairs is at most $|S|\Delta(\Delta - 1) \leq \Delta^2|S|$. The third error term comes from those pairs (x, x') for which x or x' is a neighbor of an element of E^1_H. Finally we have the pairs for which $\{x, x'\} \in \mathrm{Bad}_t$. The number of these pairs is at most $2t\Delta\varepsilon' N \leq 2\Delta\varepsilon' N^2$.

Next we will use the Cauchy-Schwarz inequality with $m = \varepsilon'' N$ and the variables $X_k, k = 1, \ldots, N$ are going to correspond to $deg_{U_t}(v)$, $v \in V_i$ (and the first m variables to degrees in F_0). Then we have

$$|D| = \varepsilon'' \sum_{v\in V_i} deg_{U_t}(v) - \sum_{v\in F_0} deg_{U_t}(v)$$

$$\geq \varepsilon'' \sum_{v\in V_i} deg_{U_t}(v) - \varepsilon''(1-\varepsilon'')d(U_t)|S|N = (\varepsilon'')^2 d(U_t)|S|N. \tag{8}$$

Then using (6), (8) and the Cauchy-Schwarz inequality we get

$$\sum_{x\in S}\sum_{x'\in S}|N_{U_t}(x)\cap N_{U_t}(x')| = \sum_{v\in V_i}(deg_{U_t}(v))^2$$

$$\geq \frac{1}{N}\left(\sum_{v\in V_i} deg_{U_t}(v)\right)^2 + (\varepsilon'')^3 d(U_t)^2 N|S|^2$$

$$\geq \frac{1}{N}\left(\sum_{x\in S}(d-\varepsilon)^{\nu(t,x)}N - \Delta r^2\varepsilon N^2\right)^2 + (\varepsilon'')^3 d(U_t)^2 N|S|^2$$

$$\geq \sum_{x\in S}\sum_{x'\in S}(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}N - 2\Delta\varepsilon' N^3 + (\varepsilon'')^3(d-\varepsilon)^{2\Delta}N|S|^2,$$

which is a contradiction with (7), since $|S| \geq (d''')^2 N$,

$$\left((d+\varepsilon)^{\nu(t,x)+\nu(t,x')} - (d-\varepsilon)^{\nu(t,x)+\nu(t,x')}\right) \ll \Delta\varepsilon,$$

and

$$(\varepsilon'')^3(d-\varepsilon)^{2\Delta}(d''')^2 \gg \frac{d^{2\Delta}}{2}(d''')^2(\varepsilon'')^3 \geq \Delta\varepsilon' \gg \Delta\varepsilon,$$

by the choice of the parameters.

An easy consequence of Lemma 2 is the following lemma.

Lemma 3. *In Step 3 we have $|E^2_i| \leq \varepsilon'' N$ for every $1 \leq i \leq r$.*

Proof. Indeed applying Lemma 2 with $t = T_0$ and $S = B_i$ (so we have $|S| = |B_i| = d'N > (d''')^2 N$) we get

$$(1 - \varepsilon'')d(U_t)|S| \geq \frac{d^\Delta}{2}|S| > d''|S|,$$

and $E_i^2 \subset F$.
From this we can prove Lemma 2 for $t > T_0$ with ε''' instead of ε''.

Lemma 4. *We are given integers $1 \leq i \leq r$, $T_0 < t \leq T$ and a set $S \subset X_i$ of unembedded vertices at time t with $|S| \geq (d''')^2|X_i| = (d''')^2 N$. If we assume that Phase 1 succeeded for all time t' with $t' \leq t$, then apart from an exceptional set F of size at most $\varepsilon''' N$, for every vertex $v \in V_i$ we have the following*

$$deg_{U_t}(v) = |\{x : x \in S, v \in C_{t,x}\}| \geq (1 - \varepsilon''')d(U_t)|S| \quad \left(\geq \frac{d^\Delta}{2}|S|\right).$$

Proof. We only have to pay attention to the neighbors of the elements of E_H^2, otherwise the proof is the same as the proof of Lemma 2 with ε''' instead of ε''. In (6) the error term becomes $\Delta r \varepsilon'' N^2$, coming from the neighbors of elements of E_H^2. In (7) we have more bad pairs, namely all pairs (x, x') where x or x' is a neighbor of an element of E_H^2. These give an additional error term of $2\Delta r \varepsilon''|S|N^2$. However, the contradiction still holds, since

$$(\varepsilon''')^3(d - \varepsilon)^{2\Delta}(d''')^2 \gg \frac{d^{2\Delta}}{2}(d''')^2(\varepsilon''')^3 \geq \Delta\varepsilon'',$$

by the choice of the parameters.
An easy consequence of Lemmas 2 and 4 is the following lemma.

Lemma 5. *We are given integers $1 \leq i \leq r$, $1 \leq t \leq T$, a set $S \subset X_i$ of unembedded vertices at time t with $|S| \geq d'''|X_i| = d''' N$ and a set $A \subset V_i$ with $|A| \geq d'''|V_i| = d''' N$. If we assume that Phase 1 succeeded for all time t' with $t' \leq t$, then apart from an exceptional set S' of size at most $(d''')^2 N$, for every vertex $x \in S$ we have the following*

$$|A \cap C_{t,x}| \geq \frac{|A|}{2N}|C_{t,x}|. \tag{9}$$

Proof. Assume indirectly that the statement is not true, i.e. there exists a set $S' \subset S$ with $|S'| > (d''')^2 N$ such that for every $x \in S'$ (9) does not hold. Once again we consider the bipartite graph $U_t = U_t(S', V_i)$. We have

$$\sum_{v \in A} deg_{U_t}(v) = \sum_{x \in S'} |A \cap C_{t,x}| < \frac{|A|}{2N} \sum_{x \in S'} |C_{t,x}| = \frac{|A|}{2N}\, d(U_t)|S'|N.$$

On the other hand, applying Lemmas 2 or 4 for S' we get

$$\sum_{v \in A} deg_{U_t}(v) \geq (1-\varepsilon'')d(U_t)|S'|(|A| - \varepsilon'' N)$$

contradicting the previous inequality.
Finally we have

Lemma 6. *For every $1 \leq t \leq T$ and for every vertex y that is unembedded at time t, if we assume that Phase 1 succeeded for all time t' with $t' \leq t$, then we have the following at time t*

$$|H_{t,y}| > d'' N. \tag{10}$$

Proof. We apply Lemma 5 with S_t the set of all unembedded vertices in X_i at time t, and $A_t = V_i \setminus Z_t$ (all uncovered vertices). Then for all but at most $(d''')^2 N$ vertices $x \in S_t$ using (2) and (4) we get

$$|H_{t,x}| = |A_t \cap C_{t,x}| \geq \frac{|A_t|}{2N}|C_{t,x}| \geq \frac{d'}{4}\delta(d-\varepsilon)^{\Delta} N \geq (d')^2 N, \tag{11}$$

if $|A_t| \geq (d'/2)N$. We will show next that in fact for $1 \leq t \leq T$, we have

$$|A_t| \geq |A_T| \geq (d'-d'')N \quad \left(\geq \frac{d'}{2}N\right),$$

so (11) always holds. Assume indirectly that this is not the case, i.e. there exists a $1 \leq T' < T$ for which,

$$|A_{T'}| \geq (d'-d'')N \;\text{ but }\; |A_{T'+1}| < (d'-d'')N.$$

From the above at any given time t for which $T_1|t$ and $1 \leq t \leq T'$, in Step 4 we find at most $(d''')^2 N$ exceptional vertices in X_i. Hence, altogether we find at most

$$\frac{1}{d''}(d''')^2 N \ll d'' N$$

exceptional vertices in X_i up to time T'. However, this implies that at time T' we still have many more than $(d'-d'')N$ unembedded buffer vertices in X_i, which in turn implies that $|A_{T'+1}| \gg (d'-d'')N$, a contradiction. Thus we have

$$|A_T| \geq (d'-d'')N, \;\; T \leq rN - rd'N + rd''N,$$

at time T (or in Phase 2) we have at least $(d'-d'')N$ unembedded buffer vertices in each X_i, and furthermore, for every $1 \leq t \leq T$ for all but at most $(d''')^2 N$ vertices $x \in S_t$ we have

$$|H_{t,x}| > (d')^2 N.$$

Let us pick an arbitrary $1 \leq t \leq T$ and an unembedded y at time t (with $\psi(y) = V_i$). We have to show that (10) holds. Let $kd''n = kT_1 \leq t < (k+1)T_1$ for some $0 \leq k \leq T/T_1$. We distinguish two cases:
Case 1. y was not among the at most $(d''')^2 N$ exceptional vertices of X_i found in Step 4 at time kT_1. Then

$$|H_{t,y}| \geq \left(\frac{\delta}{2}(d-\varepsilon)^{\Delta}(d')^2 - rd'' \right) N.$$

Indeed, at time kT_1 we had $|H_{kT_1,y}| \geq (d')^2 N$. Until time t, $H_{t,y}$ could have been cut at most once to a $\geq (\delta/2)$-fraction (if y is a neighbor of an element of E_H, there can be at most one such E_H-neighbor) and at most Δ times to a $\geq (d-\varepsilon)$-fraction (using (1) and (5)), and precisely $t - kT_1 \leq T_1 = rd''N$ new vertices were covered.
Case 2. y was among the at most $(d''')^2 N$ exceptional vertices of X_i found in Step 4 at time kT_1. Then

$$|H_{t,y}| \geq \left(\frac{\delta}{2}(d-\varepsilon)^{\Delta}(d')^2 - rd'' - r(d''')^2 \right) N,$$

since at time $(k-1)T_1$ (we certainly must have $k \geq 2$), y was not exceptional, and because the exceptional vertices were brought forward we have $t \leq kT_1 + r(d''')^2 N$. Thus in both cases we have $|H_{t,y}| > d''N$, as desired.
Finally we show that the selection algorithm always succeeds in selecting an image $\varphi(x_t)$.

Lemma 7. *For every $1 \leq t \leq T$, if we assume that Phase 1 succeeded for all time t' with $t' \leq t$, then Phase 1 succeeds for time t.*

Proof. We only have to consider Case 1 in the selection algorithm. We choose a vertex $v \in H_{t-1,x_t}$ as the image $\varphi(x_t)$ which satisfies (1), (2) and (3). We have by Lemma 6,

$$|H_{t-1,x_t}| \geq d''N.$$

By ε-regularity we have at most $2\varepsilon N$ vertices in H_{t-1,x_t} which do not satisfy (1) and similarly for (2). For (3) we define an auxiliary bipartite graph B as follows. One color class W_1 is the vertices in H_{t-1,x_t} and the other class W_2 is the sets $C_{t-1,y} \cap C_{t-1,y'}$ for all pairs $\{y, y'\}$ where $\{x_t, y\} \in E(H)$, $\psi(y) = \psi(y')$, and $\{y, y'\} \notin \mathrm{Bad}_{t-1}$. We put an edge between a $v \in W_1$ and an $S \in W_2$ if inequality (3) is not satisfied for v and S. Let us assume indirectly that we have more than $\varepsilon' N$ vertices $v \in W_1$ with $deg_B(v) > \varepsilon' |W_2|$. Then there must exist a $S \in W_2$ with

$$deg_B(S) > \varepsilon' |W_1| \gg \varepsilon N.$$

However, this is a contradiction with ε-regularity since

$$|S| \geq (d-\varepsilon)^{2\Delta} N \gg \varepsilon N.$$

Here we used the fact that the pair corresponding to S is not in Bad_{t-1}. Thus altogether we have at most $4\varepsilon N + \varepsilon' N \ll d'' N$ vertices in H_{t-1,x_t} that we cannot choose and thus the selection algorithm always succeeds in selecting an image $\varphi(x_t)$, proving Claim 1.

Proof of Claim 2. We want to show that we can find a system of distinct representatives of the sets $H_{T,x_j}, T < j \leq n$, where the sets H_{T,x_j} belong to a given cluster V_i.
To simplify notation, let us denote by Y the set of remaining vertices in V_i, and by X the set of remaining unembedded (buffer) vertices assigned to V_i. If $x = x_j \in X$ then write H_x for its possible location H_{T,x_j} at time T. Also write $M = |X| = |Y|$.
The König-Hall condition for the existence of a system of distinct representatives obviously follows from the following three conditions:

$$|H_x| > d''' M \quad \text{for all} \quad x \in X, \tag{12}$$

$$|\bigcup_{x\in S} H_x| \geq (1-d''')M \quad \text{for all subsets} \quad S \subset X, |S| \geq d''' M, \tag{13}$$

$$|\bigcup_{x\in S} H_x| = M \quad \text{for all subsets} \quad S \subset X, |S| \geq (1-d''')M. \tag{14}$$

Equation (12) is an immediate consequence of Lemma 6, (13) is a consequence of Lemma 2. Finally to prove (14), we have to show that every vertex in $Y \subset V_i$ belongs to at least $d'''|X|$ location sets H_x. However, this is trivial from the construction of the embedding algorithm, in Step 3 of Phase 1 we took care of the small number of exceptional vertices for which this is not true. This finishes the proof of Claim 2 and the proof of correctness.

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